HOMOTOPY PULLBACK OF A_n-SPACES AND ITS APPLICATIONS TO A_n-TYPES OF GAUGE GROUPS

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Abstract. We construct the homotopy pullback of A_n-spaces and show some universal property of it. As an application, we investigate A_n-types of gauge groups. In particular, we give a new result on A_n-types of the gauge groups of principal SU(2)-bundles over S^4.

1. Introduction

The concept of *A_n-space* was first introduced by Stasheff [Sta63a] as an H-space with higher homotopy associativity. In [Sta63b], he also defined morphism between them, called *A_n-homomorphism*, but, as pointed out by himself, it is too restrictive class. Later, more general morphism between them were formulated by Boardaman and Vogt [BV73] and by Iwase and Mimura [IM89]. Before their formulation, to construct mixing of A_n-types, "homotopy pullback of A_n-maps" was considered by Zabrodsky [Zab70] and by Mimura, Nishida and Toda [MNT71] in certain sense. It is shown by Iwase and Mimura [IM89] that the homotopy pullback of A_n-maps becomes an A_n-space, using the A_n-structures.

In this paper, we construct the homotopy pullback of A_n-maps as an A_n-space by giving an A_n-form (Section 4). Moreover, when the maps are A_n-homomorphisms, we show that the homotopy pullback has some universal property (Section 3).

As an application of homotopy pullback of A_n-spaces, we investigate A_n-types of gauge groups. For a principal G-bundle P, the *gauge group* $\mathcal{G}(P)$ is the topological group consisting of self-isomorphisms on P. If principal G-bundles P and P' are isomorphic, then the gauge groups $\mathcal{G}(P)$ and $\mathcal{G}(P')$ are isomorphic as topological groups. Considering the converse of this statement, if we replace 'isomorphic as topological groups' by 'homotopy equivalent as topological spaces', then it becomes much far from true. This was first pointed out by Kono's result [Kon91]. As is well known, the principal SU(2)-bundles over the four dimensional sphere S^4 is classified by the homotopy group $\pi_4(B\mathrm{SU}(2)) \cong \boldsymbol{Z}$ of the classifying space. Kono's result says that there are only 6 homotopy types of the gauge groups of them. More generally, for a compact connected Lie group G and a finite CW complex B, Crabb and Sutherland [CS00] has shown that the number of homotopy types of the gauge groups of principal G-bundles over B is finite. This problem is generalized to the equivalence relation of gauge groups which concerns their multiplicative structures. Kishimoto and Kono [KK10] first considered the condition for that the adjoint group bundle $\mathrm{ad}\, P$ (see Section 8) is trivial as a fiberwise A_n-space, of which the space of sections $\Gamma(\mathrm{ad}\, P)$ is naturally isomorphic to the gauge group $\mathcal{G}(P)$. The author [Tsu12a] has shown that the result of Crabb and Sutherland remains true even if 'homotopy types' is replaced by 'A_n-equivalence types' for $n < \infty$ (the case when $n = 2$ had been already known by them). He [Tsu12a], [Tsu12b] also considered the classification of A_n-types of gauge groups of principal SU(2)-bundles over S^4.

Though there are many complete results on classifications of gauge groups, for example, [Kon91], [HK06], [HK07], [KKKT07], [HKK08], most of them were shown by observing the order of the Samelson product and using [HK06, Lamma 3.2], which states that in some good situation, two gauge groups

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are homotopy equivalent if their p-localizations are homotopy equivalent for each prime p. The second aim of this paper is to generalize this statement for A_n-types of gauge groups as follows (Section 8).

Theorem 1.1. *Let G be a compact connected Lie group, of which the rationalization $G_{(0)}$ is homotopy equivalent to the product $S^{2n_1-1}_{(0)} \times \cdots \times S^{2n_\ell-1}_{(0)}$ of rationalized spheres. Fix a map $\epsilon : S^{r-1} \to G$ with $r \geq 2n_\ell$. For an integer $k \in \boldsymbol{Z}$, denote the principal G-bundle over S^r with classifying map $k\epsilon : S^{r-1} \to G$ by P_k. Take an integer $N \in \boldsymbol{Z}$ with the adjoint group bundle $\mathrm{ad}\, P_N$ is A_n-trivial. Then, the gauge groups $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n-equivalent if $(N,k) = (N,k')$, where (a,b) represents the greatest common divisor between a and b.*

We note that when $n = 1$, it is easy to see that $\mathrm{ad}\, P_N$ is fiberwise homotopy equivalent to the trivial bundle if and only if the Samelson product $\langle \epsilon, \mathrm{id}_G \rangle \in [S^{r-1} \wedge G, G]_0$ is annihilated by N. To generalize the homotopy version to the above A_n-version, in some sense, we need to control the A_n-form of the homotopy pullback.

Using the above theorem, we obtain the following new result:

Theorem 1.2. *Denote the principal $\mathrm{SU}(2)$-bundle over S^4 with second Chern number $k \in \boldsymbol{Z}$ by P_k. Then, for each positive integer n, there exists the positive integer a_n^{fw} such that for any $k, k' \in \boldsymbol{Z}$, the gauge groups $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n-equivalent if $(a_n^{\mathrm{fw}}, k) = (a_n^{\mathrm{fw}}, k')$. Moreover, if k is odd, the converse is also true.*

We do not determine a_n^{fw} in the present paper. But we will give some estimates on p-exponents of a_n^{fw} (Section 8). In particular, the 3-exponent will be completely determined.

We will always work in the category of compactly generated spaces because we would like to use various exponential laws.

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2. A_n-spaces and A_n-maps

Let us denote the *i-th associahedron* by $\mathcal{K}_i$ $(i = 2, 3, \cdots)$ and the *i-th multiplihedron* by $\mathcal{J}_i$ $(i = 1, 2, \cdots)$. They are used to study A_n-spaces by Stasheff [Sta63a], and by Iwase and Mimura [IM89]. There are boundary maps and degeneracy maps of them:

$$\begin{aligned}
\partial_k(r,s) &: \mathcal{K}_r \times \mathcal{K}_s \to \mathcal{K}_{r+s-1} && (1 \leq k \leq r),\\
s_k &: \mathcal{K}_i \to \mathcal{K}_{i-1} && (1 \leq k \leq i),\\
\delta_k(r,s) &: \mathcal{J}_r \times \mathcal{K}_s \to \mathcal{J}_{r+s-1} && (1 \leq k \leq r),\\
\delta(t, r_1, \cdots, r_t) &: \mathcal{K}_t \times \mathcal{J}_{r_1} \times \cdots \times \mathcal{J}_{r_t} \to \mathcal{J}_{r_1+\cdots+r_t}, &&\\
d_k &: \mathcal{J}_i \to \mathcal{J}_{i-1} && (1 \leq k \leq i),
\end{aligned}$$

which satisfy some relations, see [Sta63a] and [IM89]. Define the boundaries $\partial\mathcal{K}_i \subset \mathcal{K}_i$ and $\partial\mathcal{J}_i \subset \mathcal{J}_i$ as the unions of the images of boundary maps. Then there are homeomorphisms $(\mathcal{K}_i, \partial\mathcal{K}_i) \cong (D^{i-2}, S^{i-3})$ and $(\mathcal{J}_i, \partial\mathcal{J}_i) \cong (D^{i-1}, S^{i-2})$, where D^i is the i-dimensional disk and $S^{i-1} \subset D^i$ is its boundary sphere. For details, see [Sta63a] and [IM89].

Now, we recall the definitions of A_n-spaces and A_n-maps. **We always assume that every pointed space are well-pointed and pointed homotopy equivalent to a CW complex**.

Definition 2.1. Let X be a pointed space. A family of maps $\{m_i : \mathcal{K}_i \times X^i \to X\}_{i=2}^n$ is said to be an *A_n-form* on X if the following conditions are satisfied:

(1) $m_{r+s-1}(\partial_k(r,s)(\rho,\sigma); x_1, \cdots, x_{r+s-1}) = m_r(\rho; x_1, \cdots, x_{k-1}, m_s(\sigma; x_k, \cdots, x_{k+r-1}), x_{k+r}, \cdots, x_{r+s-1})$
(2) $m_{i-1}(s_k(\rho); x_1, \cdots, x_{i-1}) = m_i(\rho; x_1, \cdots, x_{k-1}, *, x_k, \cdots, x_{i-1})$

The pair $(X, \{m_i\})$ of a pointed space and its A_n-form is called an *A_n-space*.

Example 2.2. Let G be a topological monoid. Then G is an A_∞-space with A_∞-form $\{m_i\}$ defined by

$$m_i(\rho; x_1, \cdots, x_i) = x_1 \cdots x_i.$$

We call $\{m_i\}$ the *canonical A_∞-form* of G. Unless otherwise stated, we regard a topological monoid as an A_∞-space with the canonical A_∞-form.

Definition 2.3. Let $(X, \{m_i\})$ and $(X', \{m'_i\})$ be A_n-spaces and $f : X \to X'$ a pointed map. A family of maps $\{f_i : \mathcal{J}_i \times X^i \to X'\}_{i=1}^n$ is said to be an *A_n-form* on f if the following conditions are satisfied:

(1) $f_1 = f$,
(2) $f_{r+s-1}(\delta_k(r,s)(\rho,\sigma); x_1, \cdots, x_{r+s-1})$
$= f_r(\rho; x_1, \cdots, x_{k-1}, m_s(\sigma; x_k, \cdots, x_{k+r-1}), x_{k+r}, \cdots, x_{r+s-1})$
(3) $f_{r_1+\cdots+r_t}(\delta(t, r_1, \cdots, r_t)(\tau, \rho_1, \cdots, \rho_t); x_1, \cdots, x_{r_1+\cdots+r_t})$
$= m'_t(\tau; f_{r_1}(\rho_1; x_1, \cdots, x_{r_1}), \cdots, f_{r_t}(\rho_t; x_{r_1+\cdots+r_{t-1}+1}, \cdots, x_{r_1+\cdots+r_t}))$
(4) $f_{i-1}(d_k(\rho); x_1, \cdots, x_{i-1}) = f_i(\rho; x_1, \cdots, x_{k-1}, *, x_k, \cdots, x_{i-1})$

The pair $(f, \{f_i\})$ of a pointed map and its A_n-form is called an *A_n-map*. For two A_n-maps $(f, \{f_i\}), (f', \{f'_i\}) : (X, \{m_i\}) \to (X', \{m'_i\})$, a *homotopy* between them is a continuous family of A_n-maps $(F : I \times X \to X', \{F_i : I \times \mathcal{J}_i \times X^i \to X'\})$ parametrized by the unit interval $I = [0,1]$ such that $F_i(0, \rho; x_1, \cdots, x_i) = f_i(\rho; x_1, \cdots, x_i)$ and $F_i(1, \rho; x_1, \cdots, x_i) = f'_i(\rho; x_1, \cdots, x_i)$.

Remark 2.4. In some literature, an A_n-map is defined as a map which admits some A_n-form. But we always consider A_n-maps together with A_n-forms.

If $(f, \{f_i\}) : (X, \{m_i\}) \to (X', \{m'_i\})$ is an A_n-map and a pointed map $f' : X \to X'$ is pointed homotopic to the underlying map f, then, as easily seen, f' also admits an A_n-form $\{f'_i\}$ such that $(f', \{f'_i\})$ is homotopic to $(f, \{f_i\})$.

The definition of A_n-maps is complicated and it is difficult to treat in general. So we often consider some class of A_n-maps defined more easily.

Definition 2.5. Let $(X, \{m_i\})$ and $(X', \{m'_i\})$ be A_n-spaces. A pointed map $f : X \to X'$ is said to be an *A_n-homomorphism* if f satisfies

$$f(m_i(\rho; x_1, \cdots, x_i)) = m'_i(\rho; x_1, \cdots, x_i).$$

Now we see that an A_n-homomorphism admits an A_n-form. Iwase and Mimura [IM89] constructed maps $\pi_i : \mathcal{J}_i \to \mathcal{K}_i$ $(i = 2, 3, \cdots)$ such that

(1) $\pi_i \circ \delta_1(1, i) = \pi_i \circ \delta(i, 1, \cdots, 1) = \mathrm{id}_{\mathcal{K}_i}$,
(2) $\pi_{r+s-1} \circ \delta_k(r,s) = \partial_k(r,s) \circ (\pi_r \times \mathrm{id}_{\mathcal{K}_s})$,
(3) $\pi_{r_1+\cdots+r_t} \circ \delta(t, r_1, \cdots, r_t)$
$= \partial_{r_1+\cdots+r_{t-1}+1}(r_1+\cdots+r_{t-1}+1, r_t) \circ \cdots \circ (\partial_{r_1+1}(r_1+t-1, r_2) \times \mathrm{id}_{\mathcal{K}_{r_3} \times \cdots \times \mathcal{K}_{r_t}}) \circ (\partial_1(t, r_1) \times \mathrm{id}_{\mathcal{K}_{r_2} \times \cdots \times \mathcal{K}_{r_t}}) \circ$
$(\mathrm{id}_{\mathcal{K}_t} \times \pi_{r_1} \times \cdots \times \pi_{r_t})$,
(4) $\pi_i \circ d_k = s_k \circ \pi_i$.

If we denote the right hand side of the third equation by $D \circ (\mathrm{id}_{\mathcal{K}_t} \times \pi_{r_1} \times \cdots \times \pi_{r_t})$, D has a property that

$$m_{r_1+\cdots+r_t}(D(\tau, \rho_1, \cdots, \rho_t); x_1, \cdots, x_{r_1+\cdots+r_t}) = m_t(\tau; m_{r_1}(\rho_1; x_1, \cdots, x_{r_1}), \cdots, m_{r_t}(\rho_t; \cdots, x_{r_1+\cdots+r_t}))$$

for an A_n-space $(X, \{m_i\})$.

Thus, if $f : X \to X'$ is an A_n-homomorphism between A_n-spaces $(X, \{m_i\})$ and $(X', \{m'_i\})$, then the sequence of maps $\{f_i : \mathcal{J}_i \times X^i \to X'\}_{i=1}^n$ defined by

$$f_i(\rho; x_1, \cdots, x_i) = f(m_i(\pi_i(\rho); x_1, \cdots, x_i))$$

is an A_n-form of f. We call $\{f_i\}$ the *canonical A_n-form* of f. Unless otherwise stated, we regard an A_n-homomorphism as an A_n-map with the canonical A_n-form.

Example 2.6. A homomorphism $f : G \to G'$ between topological monoids is an A_∞-homomorphism. The canonical A_∞-form $\{f_i\}$ is given as

$$f_i(\rho; x_1, \cdots, x_i) = f(x_1 \cdots x_i).$$

Next, we consider the composition of A_n-maps. The composition of general A_n-maps is complicated. But the composite of an A_n-map with an A_n-homomorphism is easily considered as follows.

Definition 2.7. Let $(X, \{m_i\}), (X', \{m'_i\}), (X'', \{m''_i\})$ be A_n-spaces. If $(f, \{f_i\}) : (X, \{m_i\}) \to (X', \{m'_i\})$ is an A_n-map and $g : X' \to X''$ is an A_n-homomorphism, then we call the A_n-map$(g\circ f, \{g\circ f_i\})$ the *canonical composite* of $(f, \{f_i\})$ and g. Similarly, if $f : X \to X'$ is an A_n-homomorphism and $(g, \{g_i\}) : (X', \{m'_i\}) \to (X'', \{m''_i\})$ is an A_n-map, then we call the A_n-map $(g \circ f, \{g_i \circ (\mathrm{id}_{\mathcal{J}_i} \times f^{\times i})\})$ the *canonical composite* of f and $(g, \{g_i\})$.

We also consider equivalence between A_n-spaces.

Definition 2.8. Let $(X, \{m_i\}), (X', \{m'_i\})$ be A_n-spaces. Then an A_n-map $(f, \{f_i\}) : (X, \{m_i\}) \to (X', \{m'_i\})$ is said to be an A_n-*equivalence* if the underlying map f is a pointed homotopy equivalence. We say $(X, \{m_i\})$ and $(X', \{m'_i\})$ are A_n-*equivalent* if there exists an A_n-equivalence $(X, \{m_i\}) \to (X', \{m'_i\})$.

Proposition 2.9. *The relation A_n-equivalence is an equivalence relation between A_n-spaces.*

Remark 2.10. Iwase and Mimura gave an outline of the proof in [IM89]. We will give a proof of this proposition in Section 6 using the result of Boardman and Vogt [BV73].

For a pointed space X, let us denote the space consisting of the points $(x_1, \cdots, x_n) \in X^n$ with $x_k = *$ for some k by $X^{[n]}$. The localization of an A_n-space is also an A_n-space as follows.

Proposition 2.11. *Let $\mathcal{P}$ be a set of primes and $(X, \{m_i\})$ be a path-connected A_n-space. For a $\mathcal{P}$-localization map $\ell : X \to X_\mathcal{P}$, there exists an A_n-form $\{m^\mathcal{P}_i\}$ on $X_\mathcal{P}$ such that ℓ admits an A_n-form $\{\ell_i\}$ such that $(\ell, \{\ell_i\})$ is an A_n-map between $(X, \{m_i\})$ and $(X_\mathcal{P}, \{m^\mathcal{P}_i\})$.*

Proof. We construct the A_n-forms $\{m^\mathcal{P}_i\}$ and $\{\ell_i\}$ inductively. Put $\ell_1 = \ell$. Assume we have obtained A_{i-1}-forms $\{m^\mathcal{P}_j\}_{j=2}^{i-1}$ and $\{\ell_j\}_{j=1}^{i-1}$ as above. We define a map ℓ_i on $(\partial\mathcal{J}_i - \mathrm{Int}\delta(i, 1, \cdots, 1)) \times X^i \cup \mathcal{J}_i \times X^{[n]}$ by

$$\begin{aligned} &\ell_i(\rho; x_1, \cdots, x_i) \\ &= \begin{cases} \ell_s(\sigma; x_1, \cdots, x_{k-1}, m_t(\tau; x_k, \cdots, x_{k+s-1}), x_{k+s}, \cdots, x_i) & (\rho = \delta_k(s, t)(\sigma, \tau)) \\ m^\mathcal{P}_t(\tau; \ell_{s_1}(\sigma_1; x_1, \cdots, x_{s_1}), \cdots, \ell_{s_t}(\sigma_t; x_{s_1+\cdots+s_{i-1}+1}, \cdots, x_i)) & (\rho = \delta(t, s_1, \cdots, s_t)(\tau, \sigma_1, \cdots, \sigma_t), t < i) \\ \ell_{i-1}(d_k(\rho); x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_i) & (x_k = *) \end{cases}. \end{aligned}$$

Since $X^{[i]} \subset X^i$ has homotopy extension property and $\partial\mathcal{J}_i - \mathrm{Int}\delta(i, 1, \cdots, 1)$ is a deformation retract of $\mathcal{J}_i$, this map extends to $\ell_i : \mathcal{J}_i \times X^i \to X_\mathcal{P}$. Similarly, we define $m^\mathcal{P}_i$ on $\partial\mathcal{K}_i \times X^i_\mathcal{P} \cup \mathcal{K}_i \times X^{[n]}_\mathcal{P}$ by

$$m^\mathcal{P}_i(\rho; x_1, \cdots, x_i) = \begin{cases} m^\mathcal{P}_s(\sigma; x_1, \cdots, x_{k-1}, m_t(\tau; x_k, \cdots, x_{k+s-1}), x_{k+s}, \cdots, x_i) & (\rho = \partial_k(s, t)(\sigma, \tau)) \\ m^\mathcal{P}_{i-1}(s_k(\rho); x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_i) & (x_k = *) \end{cases}.$$

By the weak homotopy equivalence $(\ell^i)^* : \mathrm{Map}(X^i_\mathcal{P}, X_\mathcal{P}) \to \mathrm{Map}(X^i, X_\mathcal{P})$, $m^\mathcal{P}_i$ extends to $m^\mathcal{P}_i : \mathcal{K}_i \times X^i_\mathcal{P} \to X_\mathcal{P}$ such that

$$\ell_i(\delta(i, 1, \cdots, 1)(\rho, *, \cdots, *); x_1, \cdots, x_i) = m^\mathcal{P}_i(\rho; \ell(x_1), \cdots, \ell(x_i)).$$

Therefore, $(\ell, \{\ell_j\}_{j=1}^i)$ is an A_i-map between $(X, \{m_j\}_{j=2}^i)$ and $(X_\mathcal{P}, \{m^\mathcal{P}_j\}_{j=2}^i)$. □

Remark 2.12. Every path-connected A_n-space $(n \geq 2)$ is nilpotent and so it has its $\mathcal{P}$-localization in the sense of [HMR75].

3. Pullback of A_n-homomorphism fibrations

From now on, we often abbreviate A_n-forms of A_n-spaces and A_n-maps by underlying spaces and maps, respectively. For example, an A_n-space $(X,\{m_i\})$ is abbreviated by X, an A_n-map $(f,\{f_i\}):(X,\{m_i\})\to(X',\{m'_i\})$ by $f:X\to X'$, a homotopy $(F,\{F_i\})$ between A_n-maps $(f,\{f_i\}),(f',\{f'_i\}):(X,\{m_i\})\to(X',\{m'_i\})$ by $F:I\times X\to X'$, and so on.

As easily checked, for A_n-homomorphisms $f_1:X_1\to X_3$ and $f_2:X_2\to X_3$, the pullback (in the topological sense) X of the diagram $X_1\xrightarrow{f_1}X_3\xleftarrow{f_2}X_2$ naturally inherits an A_n-form and the natural projections $p_1:X\to X_1$ and $p_2:X\to X_2$ are A_n-homomorphisms. If the map $f_1:X_1\to X_3$ is a Hurewicz fibration, X has the following universal property.

Theorem 3.1. *Let X_1, X_2 and X_3 be A_n-spaces and $f_1:X_1\to X_3$ and $f_2:X_2\to X_3$ be A_n-homomorphisms. In addition, suppose f_1 is a Hurewicz fibration. Denote the topological pullback of $X_1\xrightarrow{f_1}X_3\xleftarrow{f_2}X_2$ by X and the projections by $p_1:X\to X_1$ and $p_2:X\to X_2$. For an A_n-space Y and A_n-maps $g_1:Y\to X_1$ and $g_2:Y\to X_2$, if the canonical composites $f_1\circ g_1$ and $f_2\circ g_2$ are homotopic as A_n-maps through a homotopy $G_3:I\times Y\to X_3$, then there exist an A_n-map $g:Y\to X$ and a homotopy $G_1:I\times Y\to X_1$ of A_n-maps from g_1 to $p_1\circ g$, where the canonical composites $p_2\circ g$ and $f_1\circ G_1$ are equal to g_2 and G_3, respectively. Here, g and G_1 are unique up to homotopy. More precisely, if $g':Y\to X$ and $G'_1:I\times Y\to X_1$ satisfy the same condition, then there exist a homotopy $\gamma:I\times Y\to X$ of A_n-maps from g to g' and a homotopy $\Gamma_1:I\times(I\times Y)\to X_1$ of homotopies of A_n-maps from G_1 to G'_1 such that the canonical composites $p_2\circ\gamma$ and $f_1\circ\Gamma_1$ are the stationary homotopies of g_2 and G_3,respectively, and the canonical composite $p_1\circ\gamma$ is equal to $\Gamma|_{I\times(\{1\}\times Y)}$.*

$$\begin{tikzcd}
Y \arrow[dr, dotted, "g"] \arrow[drr, bend left, "g_1"] \arrow[ddr, bend right, "g_2"'] & & \\
& X \arrow[r, "p_1"] \arrow[d, "p_2"] & X_1 \arrow[d, "f_1"] \\
& X_2 \arrow[r, "f_2"] & X_3
\end{tikzcd}$$

This theorem immediately follows from the property of the topological pullback and the following lemma.

Lemma 3.2. *Let X_1, X_3 and Y be A_n-spaces and $f_1:X_1\to X_3$ be an A_n-homomorphism and a Hurewicz fibration. If $g_1:Y\to X_1$ and $g_3:Y\to X_3$ are A_n-maps such that g_3 and the canonical composite $f_1\circ g_1$ are homotopic as A_n-maps through a homotopy $G_3:I\times Y\to X_3$ of A_n-maps, then there exist a homotopy $G_1:I\times Y\to X_1$ such that the canonical composite $f_1\circ G_1$ is equal to G_3.*

Moreover, if $G'_1:I\times Y\to X_1$ satisfies the same condition, then there exists a homotopy $\Gamma_1:I\times(I\times Y)\to X$ of homotopies of A_n-maps from G_1 to G'_1 such that the canonical composite $f_1\circ\Gamma_1$ is the stationary homotopy of G_3.

$$\begin{tikzcd}
Y \arrow[r, "g_1"] \arrow[dr, "g_3"] & X_1 \arrow[d, "f_1"] \\
& X_3
\end{tikzcd}$$

Proof. Denote the A_n-forms of X_1, Y, g_1, G_3 by $\{(m_1)_i\}$, $\{(m_Y)_i\}$ $\{(g_1)_i\}$ and $\{(G_3)_i\}$, respectively. As a homotopy of maps, G_3 lifts to G_1 by the covering homotopy property of f_1. Assume G_3 lifts to a homotopy of A_{i-1}-maps $\{(G_1)_j\}_{j=1}^{i-1}$. Then define a map $(G_1)_i:(((\{0\}\times\mathcal{J}_i)\cup(I\times\partial\mathcal{J}_i))\times Y^i)\cup$

$(I \times \mathcal{J}_i \times Y^{[i]}) \to X_1$ by

$$
\begin{aligned}
&(G_1)_i(u,\rho;y_1,\cdots,y_i)\\
&= \begin{cases}
(g_1)_i(\rho;y_1,\cdots,y_i) & (u=0)\\
(G_1)_s(\sigma;y_1,\cdots,y_{k-1},(m_Y)_t(\tau;y_k,\cdots,y_{k+s-1}),y_{k+s},\cdots,y_i) & (\rho=\delta_k(s,t)(\sigma,\tau))\\
(m_1)_s(\sigma;(G_1)_{t_1}(\tau_1;y_1,\cdots,y_{t_1}),\cdots,(G_1)_{t_s}(\tau_s;y_{t_1+\cdots+t_{s-1}+1},\cdots,y_i)) & (\rho=\delta(s,t_1,\cdots,t_s)(\sigma,\tau_1,\cdots,\tau_s))\\
(G_1)_{i-1}(u,d_k(\rho);y_1,\cdots,y_{k-1},y_{k+1},\cdots,y_i) & (y_k=*)
\end{cases}.
\end{aligned}
$$

Then the composite $f_1 \circ (G_1)_i$ is equal to $(G_3)_i$ at every point where $(G_1)_i$ is defined. By the covering homotopy extension property, $(G_1)_i$ extends to $(G_1)_i : I \times \mathcal{J}_i \times Y^i \to X_1$ with $f_1 \circ (G_1)_i = (G_3)_i$. Therefore, G_3 lifts to a homotopy of A_i-maps $\{(G_1)_j\}_{j=1}^i$.

The latter half follows from a quite analogous argument. □

One can prove the universal property for homotopies as follows analogously. For homotopies $H, H' : I \times Y \to X$ of A_n-maps between f and f' and between f' and f'', respectively, let us denote the composite of homotopies H and H' by $H * H'$. More precisely, the underlying homotopy $H * H' : I \times (I \times Y) \to X$ is defined as

$$(H * H')(v,u;y) = \begin{cases} H(2v,u;y) & (0 \le v \le 1/2)\\ H'(2v-1,u;y) & (1/2 \le v \le 1)\end{cases}$$

and its A_n-form is similarly defined.

Theorem 3.3. *Set the following diagram of pullback of A_n-homomorphisms as in Theorem 3.1.*

$$
\begin{tikzcd}
X \arrow[r, "p_2"'] \arrow[d, "p_1"] & X_2 \arrow[d, "f_2"'] \\
X_1 \arrow[r, "f_1"] & X_3
\end{tikzcd}
$$

For an A_n-space Y, A_n-maps $g_1, g_1' : Y \to X_1$, $g_2, g_2' : Y \to X_2$, and homotopies of A_n-maps $h_1 : I \times Y \to X_1$ from g_1 to g_1', $h_2 : I \times Y \to X_2$ from g_2 to g_2', $G_3 : I \times Y \to X_3$ from $f_1 \circ g_1$ to $f_2 \circ g_2$ and $G_3' : I \times Y \to X_3$ from $f_1 \circ g_1'$ to $f_2 \circ g_2'$, if the composite of homotopies $(f_1 \circ h_1) * G_3'$ and $G_3 * (f_2 \circ h_2)$ are homotopic as homotopies of A_n-maps, then there exist a homotopy of A_n-maps $\gamma : I \times Y \to X$ and a homotopy of homotopies of A_n-maps $\Gamma_1 : I \times (I \times Y) \to X_1$ such that the following equalities as homotopies of A_n-maps hold:

$$
\begin{aligned}
\Gamma_1|_{\{0\}\times(I\times Y)} &= h_1,\\
\Gamma_1|_{\{1\}\times(I\times Y)} &= p_1 \circ \gamma,\\
f_1 \circ \Gamma_1|_{I\times(\{0\}\times Y)} &= G_3,\\
f_1 \circ \Gamma_1|_{I\times(\{1\}\times Y)} &= G_3'.
\end{aligned}
$$

Homotopies in this theorem are described as in the following diagram which commutes up to homotopy between homotopies of A_n-maps.

$$
\begin{tikzcd}
f_1 \circ g_1 \arrow[r, Rightarrow, "G_3"] \arrow[d, Rightarrow, "f_1\circ h_1"'] & f_2 \circ g_2 \arrow[d, Rightarrow, "f_2\circ h_2"] \\
f_1 \circ g_1' \arrow[r, Rightarrow, "G_3'"] & f_2 \circ g_2'
\end{tikzcd}
$$

4. Homotopy pullback of general A_n-maps

In the previous section, we observed the pullback of A_n-homomorphisms, one of which is a Hurewicz fibration. To generalize this construction, we show the fact that every A_n-map can be "replaced" by an A_n-homomorphism which is a Hurewicz fibration.

Let $f : X \to X'$ be a pointed map between pointed spaces. As is well-known, there is a Hurewicz fibration $\tilde{f} : \tilde{X} \to X'$ and a pointed homotopy equivalence $q : \tilde{X} \to X$ such that the following diagram commutes up to homotopy,

$$\begin{tikzcd} \tilde{X} \arrow[d, "q"'] \arrow[dr, "\tilde{f}"] & \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

where

$$\tilde{X} = \{\, (x, \ell) \in X \times X'^I \mid \ell(0) = f(x) \,\},$$
$$q(x, \ell) = x, \quad \tilde{f}(x, \ell) = \ell(1).$$

We remark that q is also a Hurewicz fibration.

Proposition 4.1. *Let $(X, \{m_i\})$ and $(X', \{m'_i\})$ be A_n-spaces, $(f, \{f_i\}) : (X, \{m_i\}) \to (X', \{m'_i\})$ an A_n-map and $\tilde{X}$, $\tilde{f}$, q as above. Then $\tilde{X}$ admits an A_n-form such that $\tilde{f}$ and q are A_n-homomorphisms and the canonical composite of q and $(f, \{f_i\})$ is homotopic to $\tilde{f}$ as an A_n-map.*

Proof. We construct maps $F_i : I \times \mathcal{J}_i \times \tilde{X}^i \to X'$ and $\tilde{m}_i : \mathcal{K}_i \times \tilde{X}^i \to \tilde{X}$ inductively as follows. Define a map $F_1 : I \times \tilde{X} \to X'$ by $F_1(u, (x, \ell)) = \ell(u)$. Suppose that we have defined the maps $\{F_j\}_{j=1}^{i-1}$ and $\{\tilde{m}_j\}_{j=2}^{i-1}$ such that

$$\begin{aligned} & F_j(u, \rho; (x_1, \ell_1), \cdots, (x_j, \ell_j)) \\ &= \begin{cases} f_j(\rho; x_1, \cdots, x_j) & (u = 0) \\ M'_j(\rho; \ell_1(1), \cdots, \ell_j(1)) & (u = 1) \\ F_s(u, \sigma; (x_1, \ell_1), \cdots, \tilde{m}_t(\tau; (x_k, \ell_k), \cdots), \cdots, (x_j, \ell_j)) & (\rho = \delta_k(s, t)(\sigma, \tau), s > 1) \\ m'_s(\sigma; F_{t_1}(u, \tau_1; (x_1, \ell_1), \cdots), \cdots, F_{t_s}(u, \tau_s; \cdots, (x_j, \ell_j))) & (\rho = \delta(s, t_1, \cdots, t_s)(\sigma, \tau_1, \cdots, \tau_s)) \\ F_{j-1}(u, s_k(\rho); (x_1, \ell_1), \cdots, (x_{k-1}, \ell_{k-1}), (x_{k+1}, \ell_{k+1}), \cdots, (x_j, \ell_j)) & ((x_k, \ell_k) = *) \end{cases} \end{aligned}$$

and

$$\tilde{m}_j(\rho; (x_1, \ell_1), \cdots, (x_j, \ell_j)) = (m_j(\rho; x_1, \cdots, x_j), (u \mapsto F_j(u, \delta_1(1, j)(*, \rho); (x_1, \ell_1), \cdots, (x_j, \ell_j)))\,),$$

where $\{M'_j\}$ is the canonical A_n-form of the identity map $X' \to X'$. Let us denote $\partial(I \times \mathcal{J}_i) = (\partial I \times \mathcal{J}_i) \cup (I \times \partial \mathcal{J}_i)$. We also define F_i on $(\partial(I \times \mathcal{J}_i) - \mathrm{Int}\,(I \times \delta_1(1, i))) \times \tilde{X}^i \cup I \times \mathcal{J}_i \times X^{[i]}$ by the above formula. Since $\partial(I \times \mathcal{J}_i) - \mathrm{Int}\,(I \times \delta_1(1, i))$ is a deformation retract of $I \times \mathcal{J}_i$ and $\tilde{X}$ is well-pointed, F_i extends over $I \times \mathcal{J}_i \times \tilde{X}^i$ and then $\tilde{m}_i$ is obtained by the above formula.

Thus $\{\tilde{m}_i\}_{i=2}^n$ is an A_n-form of $\tilde{X}$ and $\tilde{f}$ is an A_n-homomorphism. The family of maps $\{F_i\}$ is a homotopy from $\{f_i \circ (\mathrm{id}_{\mathcal{J}_i} \times q^{\times i})\}$ to $\{M'_i \circ (\mathrm{id}_{\mathcal{J}_i} \times \tilde{f}^{\times i})\}$ through A_n-forms, where $\{M'_i \circ (\mathrm{id}_{\mathcal{J}_i} \times \tilde{f}^{\times i})\}$ is the canonical A_n-form of $\tilde{f}$. □

Remark 4.2. If X and X' are topological monoids and f is a homomorphism, then $\tilde{X}$ is naturally a topological monoid and $\tilde{f}$ is homotopic to $f \circ q$ as a homomorphism.

Now we recall homotopy pullback. For a diagram $X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$, the *homotopy pullback* X of this diagram is the topological pullback of the diagram $\tilde{X}_1 \xrightarrow{\tilde{f}_1} X_3 \xleftarrow{\tilde{f}_2} \tilde{X}_2$. There are natural projections $q_i : X \to X_i$ $(i = 1, 2)$, which is defined as the composite of the canonical projections $p_i : X \to \tilde{X}_i$ and $\tilde{q}_i : \tilde{X}_i \to X_i$. If there exists the following homotopy commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_3 \arrow[d] & X_2 \arrow[l] \arrow[d] \\ Y_1 \arrow[r] & Y_3 & Y_2 \arrow[l] \end{tikzcd}$$

then there exists a lift $X \to Y$. Moreover, if the vertical arrows are homotopy equivalence, then the lift $X \to Y$ is also a homotopy equivalence.

By the previous proposition, we immediately obtain the following theorem.

Theorem 4.3. *Let $f_1 : X_1 \to X_3$ and $f_2 : X_2 \to X_3$ be A_n-maps between A_n-spaces. Then the homotopy pullback X of $X_1 \xrightarrow{f_1} X_3 \xleftarrow{f_2} X_2$ admits an A_n-form such that the natural projections $q_1 : X \to X_1$ and $q_2 : X \to X_2$ are A_n-homomorphisms and the canonical composites $f_1 \circ q_1 : X \to X_3$ and $f_2 \circ q_2 : X \to X_3$ are homotopic as A_n-maps.*

Remark 4.4. Though we have shown the universal property of homotopy pullback of A_n-homomorphisms in Theorem 3.1, the above theorem does not claim any universal property of homotopy pullback of general A_n-maps. To state the universal property, we need the "category of A_n-spaces with higher homotopy". As far as the author knows, the only realization of this is obtained by Boardman and Vogt [BV73]. But it is realized not as a category but as a *restricted Kan complex* (also called *quasi-category* [Joy02] or ∞-*category* [Lur09]). So, the universal property should be stated in the language of restricted Kan complex. But we do not consider this problem here because it is not necessary for our later application.

As an application, the result of Zabrodsky [Zab70] is shown as follows.

Theorem 4.5 ([Zab70])**.** *For any prime p, there is a finite CW complex X which admits an A_{p-1}-form but no A_p-form.*

Proof. The case when $p = 2$ is well-known. Let p be an odd prime and $\mathcal{P}$ be the set of other primes. Recall the fact that the double suspension map $\Sigma^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$ induces the isomorphisms $\pi_i(S^{2n-1})_{(p)} \to \pi_i(\Omega^2 S^{2n+1})_{(p)}$ on p-localized homotopy groups for $i < 2pn - 3$. This implies that the p-localized sphere $S^{2n-1}_{(p)}$ admits an A_{p-1}-form since $\Omega^2 S^{2n+1}_{(p)}$ is an A_∞-space.

Assume $S^{2n-1}_{(p)}$ admits an A_p-form, then there exists a homotopy fibration $S^{2n-1}_{(p)} \to S^{2pn-1}_{(p)} \to X$ by [Sta63a]. Then the cohomology ring is computed as $H^*(X; \boldsymbol{Z}/p\boldsymbol{Z}) \cong (\boldsymbol{Z}/p\boldsymbol{Z})[x]/(x^p)$, where $x \in H^{2n}(X; \boldsymbol{Z}/p\boldsymbol{Z})$ is a generator. Moreover, a space X' defined as the cofiber $S^{2pn-1}_{(p)} \to X \to X'$ has the cohomology ring $H^*(X'; \boldsymbol{Z}/p\boldsymbol{Z}) \cong (\boldsymbol{Z}/p\boldsymbol{Z})[x]/(x^{p+1})$ by the Thom-Gysin sequence. By the Adem relation of Steenrod operations, $\mathcal{P}^1\mathcal{P}^{n-1}x = nx^p$, where $\mathcal{P}^{n-1}x \in H^{2pn-2(p-1)}(X'; \boldsymbol{Z}/p\boldsymbol{Z})$. If $n = p + 1$, then $\mathcal{P}^p x \in H^{2p^2-2}(X'; \boldsymbol{Z}/p\boldsymbol{Z}) = 0$ and $(p + 1)x^p \neq 0$. This is contradiction and then implies that $S^{2p+1}_{(p)}$ admits no A_p-form.

Next, consider the homotopy pullback Y of the diagram $S^3_{(p)} \times \cdots \times S^{2p+1}_{(p)} \to K(\boldsymbol{Q}, 3) \times \cdots \times K(\boldsymbol{Q}, 2p + 1) \leftarrow \mathrm{SU}(p + 1)_{\mathcal{P}}$ of localizations of A_{p-1}-spaces, where $\mathcal{P}$ is the set of primes other than p. This is justified since $\mathrm{SU}(p + 1)_{(0)}$ is A_∞-equivalent to $K(\boldsymbol{Q}, 3) \times \cdots \times K(\boldsymbol{Q}, 2p + 1)$ (because the classifying space $B\mathrm{SU}(p+1)_{(0)}$ is homotopy equivalent to $K(\boldsymbol{Q}, 4)\times \cdots \times K(\boldsymbol{Q}, 2p+2)$) and for each i, the rationalization $S^{2i-1}_{(p)} \to K(\boldsymbol{Q}, 2i - 1)$ admits an A_{p-1}-form by an easy obstruction argument. As easily seen, Y is homotopy equivalent to a finite CW complex of which cells have the same dimensions as ones of $S^3 \times \cdots \times S^{2p-1}$ and $\mathrm{SU}(p + 1)$. By Theorem 4.3, Y admits an A_{p-1}-form. Then, since the retract $S^{2p+1}_{(p)}$ of $S^3_{(p)} \times \cdots \times S^{2p+1}_{(p)}$ admits no A_p-form, the p-localization $Y_{(p)} \simeq S^3_{(p)} \times \cdots \times S^{2p+1}_{(p)}$ admits no A_p-form. Therefore, the finite complex Y admits no A_p-form. □

5. Framed fiberwise homotopy theory

From this section to Section 7, we prepare the tools to handle fiberwise A_n-spaces for our later applications to gauge groups.

First of all, we recall the terminology of fiberwise homotopy theory [CJ98].

Definition 5.1. Let B be a space.

(1) A *fiberwise space* over B consists of a space E together with a map $\pi : E \to B$, called the *projection*.
(2) For fiberwise spaces $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B$ and a map $f : E \to E'$, f is called a *fiberwise map* over B if $\pi' \circ f = \pi$.
(3) For fiberwise spaces $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B$ and fiberwise maps $f_0, f_1 : E \to E'$, a homotopy $f : I \times E \to E'$ between f_0 and f_1 is called a *fiberwise homotopy* if $\pi' \circ f = \pi \circ p_2$, where $p_2 : I \times E \to E$ is the second projection.
(4) A *fiberwise pointed space* over B consists of a fiberwise space $E \xrightarrow{\pi} B$ together with a section $\sigma : B \to E$ of π ($\pi \circ \sigma = \mathrm{id}_B$). Moreover, we require fiberwise pointed spaces *fiberwise well-pointedness*: for a fiberwise map $f : E \to E'$, every fiberwise homotopy $h : I \times \sigma(B) \to E'$ with $h|_{\{0\}\times\sigma(B)} = f$ extends to a fiberwise homotopy $h' : I \times E \to E'$ such that $h'|_{\{0\}\times E} = f$. Note that each fiber E_b of a fiberwise pointed space E is considered as a pointed space with basepoint $\sigma(b)$.
(5) For fiberwise pointed spaces $B \xrightarrow{\sigma} E \xrightarrow{\pi} B$ and $B \xrightarrow{\sigma'} E' \xrightarrow{\pi'} B$ and a map $f : E \to E'$, f is called a *fiberwise pointed map* over B if $\pi' \circ f = \pi$ and $f \circ \sigma = \sigma'$.
(6) *Fiberwise pointed homotopies* are defined to be section-preserving fiberwise homotopies in the obvious way.
(7) For fiberwise spaces $E \xrightarrow{\pi} B$ and $E' \xrightarrow{\pi'} B$, the *fiber product* $E \times_B E'$ is defined as

$$E \times_B E' = \{\, (e, e') \in E \times E' \,|\, \pi(e) = \pi'(e') \,\}$$

with the obvious projection $E \times_B E' \to B$. We denote the n-fold fiber product of E by $E^{\times_B n}$. The fiber product of fiberwise pointed spaces is defined as a fiber product of them with obvious section.
(8) A fiberwise pointed space $B \xrightarrow{\sigma} E \xrightarrow{\pi} B$ is said to be an *ex-fibration* if it has the following *pointed homotopy lifting property*: for any base space B' and fiberwise pointed space $B' \xrightarrow{\sigma'} E' \xrightarrow{\pi'} B'$ over B', if $F : I \times B' \to B$ is a homotopy and a map $f_0 : E' \to E$ satisfies $\pi \circ f_0 = (F|_{\{0\}\times B'}) \circ \pi'$ and $f_0 \circ \sigma' = \sigma \circ (F|_{\{0\}\times B'})$, there is a homotopy $f : I \times E' \to E$ covering F (i.e. $\pi \circ f = F \circ (\mathrm{id}_I \times \pi')$) such that $f|_{\{0\}\times E'} = f_0$ and $f \circ (\mathrm{id}_I \times \sigma) = \sigma \times F$.

Remark 5.2. In the rest of this section, we quote May's result in [May75]. The terminology he used are different from the above. Let $\mathcal{U}$ be the category of compactly generated spaces and $\mathcal{T}$ be the category of well-pointed compactly generated spaces. We list corresponding his terminology here.

(1) $\mathcal{U}$-space
(2) $\mathcal{U}$-map
(3) $\mathcal{U}$-homotopy
(4) $\mathcal{T}$-space
(5) $\mathcal{T}$-map
(6) $\mathcal{T}$-homotopy
(7) (He did not used fiber product in [May75].)
(8) $\mathcal{T}$-fibration

We remark that May's $\mathcal{U}$-fibration means just Hurewicz fibration.

Let us introduce framed fibrations. Since the unframed theory has already been established, we concentrate on the case that the base space is path-connected.

Definition 5.3. Fix a space X, a Hurewicz fibration $E \to B$ over a path-connected pointed space B is said to be *X-framed* if a homotopy equivalence, which we call *framing*, from X to the fiber E_* over the basepoint is given. For X-framed Hurewicz fibrations E and E' over B, a fiberwise map $f : E \to E'$ is said to be *X-framed* if the following diagram commutes up to homotopy and such homotopy h is

given:

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ E_* \arrow[rr, "f"] & & E'_* \end{tikzcd}$$

where $X \to E_*$ and $X \to E'_*$ are the framings. Two X-framed fiberwise spaces are said to be *X-framed equivalent* if there exists a X-framed fiberwise map between them.

Similarly, an ex-fibration $E \to B$ over a path-connected pointed space B is said to be *X-framed* if a pointed homotopy equivalence from X to the fiber E_* over the basepoint is given. The maps and equivalences between them are also similarly defined.

Remark 5.4. By Dold's Theorem [May75, Theorem 2.6], X-framed equivalence is an equivalence relation between X-framed Hurewicz or ex-fibrations.

Let B be a path-connected pointed space. For a space X with the homotopy type of a CW complex, define a set $\mathcal{E}X(B)_0$ as the set of X-framed equivalence classes of X-framed Hurewicz fibrations over B. Similarly, for a pointed space X pointed homotopy equivalent to a CW complex, define a set $\mathcal{E}_0X(B)_0$ as the set of X-framed equivalence classes of X-framed ex-fibrations over B. Let $\varphi : B \to B'$ be a pointed map. Then the pullback of fibrations by φ induces maps $\mathcal{E}X(B')_0 \to \mathcal{E}X(B)_0$ and $\mathcal{E}_0X(B')_0 \to \mathcal{E}_0X(B)_0$. This map is determined by a pointed homotopy class of φ. This follows from the framed versions of [May75, Lemma 2.4], which is shown by the similar argument but we should pay attention to that the well-pointedness of base spaces is needed. The following classification theorem is also shown similarly to the unframed versions [May75, Theorem 9.2].

Theorem 5.5 (The classification theorem for framed fibrations)**.** *The following statements hold.*

(1) *Let X be a space with the homotopy type of a CW complex. Then there exists an X-framed Hurewicz fibration EX over the classifying space BFX of the topological monoid FX of self homotopy equivalences on X such that the map $\Phi : [B, BFX]_0 \to \mathcal{E}X(B)_0$ given by the pullback $[\varphi] \mapsto [\varphi^* EX]$ is an isomorphism for any pointed CW complex B.*

(2) *Let X be a pointed space pointed homotopy equivalent to a CW complex. Then there exists an X-framed ex-fibration EX over the classifying space BHX of the topological monoid HX of self pointed homotopy equivalences on X such that the map $\Phi : [B, BHX]_0 \to \mathcal{E}_0X(B)_0$ given by the pullback $[\varphi] \mapsto [\varphi^* EX]$ is an isomorphism for any pointed CW complex B.*

Proof. Define a map $\Psi : \mathcal{E}X(B)_0 \to [B, BFX]_0$ by the following procedure. For a X-framed Hurewicz fibration $E \to B$, we have the associated principal fibration

$$PE = \{\, u : X \to E \mid u \,\text{is a homotopy equivalence between}\, X \,\text{and some fiber}\,\} \longrightarrow B$$

with basepoint given by the framing $X \to E$ and the following diagram of principal quasi-fibrations.

$$\begin{tikzcd} PE \arrow[d] & B(PE, FX, FX) \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & B(*, FX, FX) = EFX \arrow[d] \\ B \arrow[rr, bend right, "\psi"'] & B(PE, FX, *) \arrow[l, "\simeq"'] \arrow[r] & B(*, FX, *) = BFX \end{tikzcd}$$

The horizontal arrows are natural maps and the map ψ is the composite of the homotopy inverse of $B(PE, FX, *) \to B$ and the map $B(PE, FX, *) \to BFX$. Define $\Psi(E) = [\psi]$. Since our framed maps preserve framing only up to homotopy, the well-definedness of Ψ is not so trivial as in May's original proof. More precisely, if there is an X-framed fibrewise map $(f, h) : E \to E'$, $B(Pf, \mathrm{id}_{FX}, *) : B(PE, FX, *) \to B(PE', FX, *)$ is not necessarily a pointed map, so, to verify $\Psi(E) = \Psi(E')$, we need to deform $Pf : PE \to PE'$ to a pointed FX-equivariant map by using the homotopy h. Once the

well-definedness is established, the theorem follows from the completely same argument as in May's proof.

The latter half also follows from the analogous argument. □

Remark 5.6. In the corollaries of [May75, Theorem 9.2], the fibers are assumed to be compact. But this assumption is not essential. See the addenda of [May75] and [May80, Lemma 1.1].

Remark 5.7. As [May75, Theorem 9.2], we can also show the "framed fiber bundle" version.

6. Quasi-category of fiberwise A_n-spaces

In this section, we recall the definition of fiberwise A_n-spaces and introduce the framed version.

Definition 6.1. Let E be an ex-fibration over B. A family of fiberwise maps $\{m_i : \mathcal{K}_i \times E^{\times_B i} \to E\}_{i=2}^n$ over B is said to be a *fiberwise A_n-form* on E if it restricts to an A_n-form $\{m_i : \mathcal{K}_i \times (E_b)^i \to E_b\}_{i=2}^n$ on each fiber E_b. An ex-fibration equipped with a fiberwise A_n-form on it is called a *fiberwise A_n-space*. Fiberwise A_n-maps, their fiberwise homotopies, fiberwise A_n-homomorphisms and canonical composites are defined similarly.

We note that every A_n-space can be considered as a fiberwise A_n-space over a point.

Boardman and Vogt [BV73] constructed the quasi-categories (the term used in [Joy02]), which they called restricted Kan complexes, consisting of their "$W\mathfrak{B}$-spaces".

Definition 6.2. Let $\mathcal{R} = \{\mathcal{R}_i\}_{i=0}^\infty$ be a simplicial class. Then $\mathcal{R}$ is said to be a *quasi-category* if the following *restricted Kan condition* holds for $i \geq 2$: for any $i-1$-simplices $x_0, \cdots, x_{j-1}, x_{j+1}, \cdots, x_i \in \mathcal{R}_{i-1}$, $0 < j < i$, such that $d^{\ell-1}x_k = d^k x_\ell$ for $0 \leq k < \ell \leq i$ and $k, \ell \neq j$, there exists an i-simplex $x \in \mathcal{R}_i$ such that $d^k x = x_k$ for $k \neq j$.

For a quasi-category $\mathcal{R}$, the *fundamental category* $\mathrm{ho}\mathcal{R}$ of $\mathcal{R}$ is the category whose objects are $\mathcal{R}_0$ and the morphisms between $a, b \in \mathcal{R}_0$ are the simplicial homotopy classes of edges $f \in \mathcal{R}_1$ with $d^1 f = a$ and $d^0 f = b$, where the composite of $[f] : a \to b$ and $[g] : b \to c$ is defined to be such a map $[h] : a \to c$ as there exists a 2-simplex $\sigma \in \mathcal{R}_2$ with $d^2\sigma = f$, $d^0\sigma = g$ and $d^1\sigma = h$.

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "h"] & \\ b \arrow[rr, "g"] & & c \end{tikzcd}$$

Remark 6.3. In the category $\mathrm{ho}\mathcal{R}$, the identity of $a \in \mathcal{R}$ is represented by $s^0 a \in \mathcal{R}_1$.

Now we recall the terminology and results of [BV73] arranged for fiberwise A_n-spaces as explained in [Tsu12a, Section 3 and 4]. But now we need to use the pointed version.

(1) Fiberwise A_n-spaces and fiberwise $Q^n h\mathfrak{A}$-maps.

Let $\mathfrak{A}(n,1)$ be a point and then $\mathfrak{A}$ has the unique *based monochrome PRO* [BV73, Section V.2] structure. Applying the *W''-construction* [BV73, Section V.3] to $\mathfrak{A}$, we obtain the associahedron $(W''\mathfrak{A})(n,1) \cong \mathcal{K}_n$ for each $n \geq 2$ and $W''\mathfrak{A}(0,1)$ and $W''\mathfrak{A}(1,1)$ are one point spaces, where the boundary map $\partial_k(r,s) : \mathcal{K}_r \times \mathcal{K}_s \to \mathcal{K}_{r+s-1}$ and the degeneracy map $s_k : \mathcal{K}_i \to \mathcal{K}_{i-1}$ on associahedra correspond to the composite s-ary tree on the k-th twig and the composite the stump on the k-th twig in $W''\mathfrak{A}$, respectively. We denote the based PRO-subcategory of $W''\mathfrak{A}$ generated by $(W''\mathfrak{A})(i,1)$ for $i \leq n$ by $Q^n W''\mathfrak{A}$. Then, fiberwise $Q^n W''\mathfrak{A}$-spaces, defined similarly to [BV73, Definition 5.1], and our fiberwise A_n-spaces coincide. The linear category $\mathfrak{L}_n = \{0 \to 1 \to \cdots \to n\}$ is regarded as a PRO colored by the set $\{0, 1, \cdots, n\}$. Consider the Boardman-Vogt tensor product [BV73, Section II.3] $\mathfrak{A} \otimes \mathfrak{L}_n$ of $\mathfrak{A}$ and $\mathfrak{L}_n$, which is a based PRO colored by $\{0, 1, \cdots, n\}$. Similarly to $Q^n W''\mathfrak{A}$, let $Q^n HW''(\mathfrak{A} \otimes \mathfrak{L}_m)$ be the homogeneous PRO-subcategory of $HW''(\mathfrak{A} \otimes \mathfrak{L}_m)$ generated by $HW''(\mathfrak{A} \otimes \mathfrak{L}_m)(\underline{i}, k)$, $\underline{i} : [r] \to \mathrm{ob}\,\mathfrak{L}_m$ ($[r] = \{1, \cdots, r\}$) with $r \leq n$. We call a fiberwise $Q^n HW''(\mathfrak{A} \otimes \mathfrak{L}_1)$-space a *fiberwise $Q^n h\mathfrak{A}$-map*. Further, if its underlying map is a fiberwise pointed homotopy equivalence, it is said to be

a *fiberwise* $Q^n h\mathfrak{A}$*-equivalence*. Similarly, we use the terms $Q^n h\mathfrak{A}$*-map* and $Q^n h\mathfrak{A}$*-equivalence* between A_n-spaces.

(2) Correspondence of fiberwise A_n-maps and fiberwise $Q^n h\mathfrak{A}$-maps.

In a similar manner to [BV73, Definition 4.1], one can define the based PRO subcategory $LW''(\mathfrak{A}\otimes\mathfrak{L}_1) \subset HW''(\mathfrak{A}\otimes\mathfrak{L}_1)$ consisting of level-trees. The PRO subcategory $Q^n LW''(\mathfrak{A}\otimes\mathfrak{L}_1) \subset LW''(\mathfrak{A}\otimes\mathfrak{L}_1)$ generated by $LW''(\mathfrak{A}\otimes\mathfrak{L}_1)(\underline{i},k)$ for $\underline{i}:[r]\to\{0,1\}$ with $r\le n$ is a deformation retract of $Q^n HW''(\mathfrak{A}\otimes\mathfrak{L}_1)$ as a based PRO subcategory by a similar argument to the proof of [BV73, Proposition 4.6]. Then there is a natural homeomorphism $LW''(\mathfrak{A}\otimes\mathfrak{L}_1)(0^i,1)\cong\mathcal{J}_i$ ($\epsilon^i:[i]\to\{0,1\}$ denotes the constant function to $\epsilon\in\{0,1\}$) for each i which has the compatibility about boundary and degeneracy maps similar to the case of associahedra explained above. Thus fibrewise $Q^n LW''(\mathfrak{A}\otimes\mathfrak{L}_1)$-spaces just correspond to fiberwise A_n-maps. These results imply that fiberwise A_n-maps and fiberwise $Q^n h\mathfrak{A}$-maps correspond one-to-one up to fiberwise homotopy preserving the multiplicative structures.

(3) Quasi-category of fiberwise A_n-spaces.

Quite analogous to [BV73, Theorem 5.23], the sequence $\mathcal{R}^n(B)=\{\mathcal{R}^n_i(B)\}_{i=0}^{\infty}$ of classes such that $\mathcal{R}^n_i(B)$ consists of all fiberwise $Q^n W''(\mathfrak{A}\otimes\mathfrak{L}_i)$-spaces over the base B is a quasi-category. The boundary map $d^k:\mathcal{R}^n_i(B)\to\mathcal{R}^n_{i-1}(B)$ is induced from the k-th injection $\mathfrak{L}_{i-1}\to\mathfrak{L}_i$ and the degeneracy map $s^k:\mathcal{R}^n_i(B)\to\mathcal{R}^n_{i+1}(B)$ is induced from the k-th surjection $\mathfrak{L}_{i+1}\to\mathfrak{L}_i$. Obviously, a map $B'\to B$ induces a simplicial map $\mathcal{R}^n(B)\to\mathcal{R}^n(B')$.

(4) Composite of fiberwise $Q^n h\mathfrak{A}$-maps.

The map $\mathcal{J}_n\cong LW''(\mathfrak{A}\otimes\mathfrak{L}_1)(0^n,1)\hookrightarrow HW''(\mathfrak{A}\otimes\mathfrak{L}_1)(0^n,1)\to W''\mathfrak{A}(n,1)\cong\mathcal{K}_n$ induced from the map $\mathfrak{L}_1\to\mathfrak{L}_0$ satisfies the same condition for π_n in Section 2. For a simplex $\sigma\in\mathcal{R}^n_i(B)$ and a fiberwise A_n-homomorphism $g:E'_0\to E_0$, define the *canonical composite* $\sigma\circ g\in\mathcal{R}^n_i(B)$ by

$$(\sigma\circ g)(\rho)(x_1,\cdots,x_j)=\begin{cases} E'_0(\rho)(x_1,\cdots,x_j) & (\rho\in Q^n HW''(\mathfrak{A}\otimes\mathfrak{L}_i)(0^j,0)) \\ \sigma(\rho)(g(x_1),\cdots,g(x_i)) & (\rho\in Q^n HW''(\mathfrak{A}\otimes\mathfrak{L}_i)(0^j,\ell),\ell\ge 1) \\ \sigma(\rho)(x_1,\cdots,x_i) & (\rho\in Q^n HW''(\mathfrak{A}\otimes\mathfrak{L}_i)(k^j,\ell),k\ge 1)\end{cases},$$

where $E'_0\in\mathcal{R}^n_0$ denotes the $Q^n W''\mathfrak{A}$-structure of E'_0. Similarly, for a fiberwise A_n-homomorphism $h:E_n\to E'_n$, the canonical composite $h\circ\sigma\in\mathcal{R}^n_i(B)$ is also defined. Then, for a fiberwise A_n-homomorphism $f:E\to E'$, the fiberwise $Q^n h\mathfrak{A}$-map $f\circ(s^0E)=(s^0E')\circ f$ corresponds to the canonical A_n-form of f. Using this, the canonical composite is compatible with the composite in the fundamental category $\mathrm{ho}\mathcal{R}^n(B)$ as follows: for a fiberwise A_n-homomorphism $g:E\to E'$ and a fiberwise $Q^n h\mathfrak{A}$-map $f:E'\to E''$, the 2-simplex $\sigma:=(s^0f)\circ g\in\mathcal{R}^n_2(B)$ satisfies $d^2\sigma=(s^0E')\circ g$, $d^0\sigma=f$ and $d^1\sigma=f\circ g$.

$$\begin{tikzcd} & E \arrow[dl, "(s^0E')\circ g"'] \arrow[dr, "f\circ g"] & \\ E' \arrow[rr, "f"] & & E'' \end{tikzcd}$$

Similarly, for a fiberwise $Q^n h\mathfrak{A}$-map $f:E\to E'$ and a fiberwise A_n-homomorphism $h:E'\to E''$, the 2-simplex $\sigma'=h\circ(s^1f)\in\mathcal{R}^n_2(B)$ satisfies $d^2\sigma'=f$, $d^0\sigma'=h\circ(s^0E')$ and $d^1\sigma'=h\circ f$.

(5) Fiberwise homotopy invariance.

Analogously to [BV73, Theorem 5.25], if E is a fiberwise A_n-space and $f:E'\to E$ is a fiberwise pointed homotopy equivalence between ex-fibrations, then there exists a fiberwise A_n-form on E' such that f admits a fiberwise A_n-form. Also, we obtain the following as the counterpart of [BV73, Theorem 5.24]: for a fiberwise $Q^n h\mathcal{A}$-equivalence $f:E\to E'$, the fiberwise homotopy inverse g of the underlying fiberwise map f also admits the structure of a fiberwise $Q^n h\mathfrak{A}$-map which represents the inverse of f in the fundamental category $\mathrm{ho}\mathcal{R}^n(B)$. This verifies Proposition 2.9. Similarly to [BV73, Lemma 5.7], for fiberwise A_n-forms $\{m_i\}$

and $\{m_i'\}$ on a ex-fibration E, $\{m_i\}$ and $\{m_i'\}$ are homotopic as fiberwise A_n-forms on E if and only if the identity map $\mathrm{id}_E : E \to E$ admits a fiberwise A_n-form as a fiberwise A_n-map $(E, \{m_i\}) \to (E, \{m_i'\})$.

(6) Fiberwise localization.

Let E be a fiberwise A_n-space over B whose fibers are path-connected and $\mathcal{P}$ a set of primes. Since the fibers of E are nilpotent, there exists a fiberwise $\mathcal{P}$-localization $\ell : E \to \bar{E}$. By an analogous argument to the proof of Proposition 2.11, there is a fiberwise $Q^n h\mathfrak{A}$-map $\lambda \in \mathcal{R}^n_1(B)$ such that $d^1\lambda = E$ and the underlying map of λ is ℓ. Moreover, for a fiberwise $Q^n h\mathfrak{A}$-map $f : E \to E'$ such that the fibers of E' are $\mathcal{P}$-local, then there exists a 2-simplex $\sigma \in \mathcal{R}^n_2(B)$ such that $d^2\sigma = \lambda$ and $d^1\sigma = f$ because the induced map $(\ell^i)^* : \mathrm{Map}^B_B(\bar{E}^i, E') \to \mathrm{Map}^B_B(E^i, E')$ between the space of pointed fiberwise maps is a weak homotopy equivalent. This universal property implies that the fiberwise $\mathcal{P}$-localization as a fiberwise A_n-space is unique up to fiberwise A_n-equivalence.

Now we consider the framed version.

Definition 6.4. A fiberwise A_n-space $E \to B$ over a path-connected pointed space B with every fiber A_n-equivalent to an A_n-space G is said to be *G-framed* if a $Q^n h\mathfrak{A}$-equivalence from G to the fiber E_* over the basepoint is given. For G-framed fiberwise A_n-spaces E and E' over B, a fiberwise $Q^n h\mathfrak{A}$-map $f : E \to E'$ is said to be *G-framed* if a 2-simplex $\sigma \in \mathcal{R}^n_2(*)$ is given and satisfies the following conditions:

(1) $d^2\sigma$ is the G-framing $G \to E_*$ of E,
(2) $d^1\sigma$ is the G-framing $G \to E'_*$ of E,
(3) $d^0\sigma$ is the restriction $f_* : E_* \to E'_*$ of f.

Two G-framed fiberwise A_n-spaces are said to be *G-framed equivalent* if there exists a G-framed fiberwise $Q^n h\mathfrak{A}$-map between them. In particular, a G-framed fiberwise A_n-space over B is said to be *A_n-trivial* if it is G-framed equivalent to the product $B \times G$ with framing $G \cong * \times G \subset B \times G$.

The relation G-framed equivalence is the equivalence relation between fiberwise A_n-spaces by Dold's theorem and the property of fiberwise A_n-spaces stated above.

7. Classification theorem for framed fiberwise A_n-spaces

The author has shown the classification theorem for fiberwise A_n-spaces [Tsu12a, Theorem 5.7], which is used to show the finiteness of fiberwise A_n-types of adjoint group bundles (see Section 8). In this section, we show the classification theorem for framed fiberwise A_n-spaces.

First, we arrange the classification theorem for our fiberwise A_n-spaces. Let G be an A_n-space. Denote the set of equivalence classes of fiberwise A_n-spaces over a path-connected well-pointed space B whose fibers are A_n-equivalent to G by $\mathcal{E}^{A_n}G(B)$. For an ex-fibration E over B whose fibers are pointed homotopy equivalent to G, define a space $M_n[E]$ as

$$M_n[E] = \coprod_{b \in B} \left\{ \{m_i\} \in \prod_{i=2}^{n} \mathrm{Map}\,(\mathcal{K}_i \times E_b^i, E_b) \;\middle|\; \begin{array}{l} \{m_i\} : \text{an } A_n\text{-form of } E_b \text{ such that} \\ E_b \text{ and } G \text{ are } A_n\text{-equivalent} \end{array} \right\},$$

which is topologized as a subspace of the fiber product of appropriate fiberwise mapping spaces over B. There is a sequence of natural projections

$$M_n[E] \longrightarrow M_{n-1}[E] \longrightarrow \cdots \longrightarrow M_2[E] \longrightarrow M_1[E] = B,$$

each of which is a Hurewicz fibration because of the homotopy extension property of the inclusion $\partial\mathcal{K}_i \hookrightarrow \mathcal{K}_i$ and the well-pointedness of G. Further, by an easy observation, we obtain the homotopy fiber sequence

$$\Omega_0^{n-2}\mathrm{Map}_0(G^{\wedge n}, G) \longrightarrow M_n(G) \longrightarrow M_{n-1}(G),$$

where $G^{\wedge n}$ denotes the n-fold smash product $G \wedge \cdots \wedge G$, $\mathrm{Map}_0(X,Y)$ represents the space of all pointed maps $X \to Y$ and $\Omega_0^m X$ represents the space of all pointed maps $S^m \to X$ homotopic to the constant map. By the exponential law, each fiberwise A_n-form on E corresponds to a section of the projection $M_n[E] \to B$. The pullback $E_n[E]$ of E by the projection $M_n[E] \to B$ has a natural fiberwise A_n-form such that the restricted A_n-form of the fiber over a point $\{m_i\} \in M_n[E]$ is $\{m_i\}$. Let $E_1(G) \to M_1(G) = BHG$ be the universal ex-fibration with the fibers pointed homotopy equivalent to G. Let us denote $M_n(G) = M_n[E_1(G)]$ and $E_n(G) = E_n[E_1(G)]$.

Using the properties of fiberwise A_n-spaces explained in the previous section, one can prove the following theorem by a completely parallel argument in [Tsu12a, Section 5]. Denote the free homotopy set between spaces X and Y by $[X,Y]$.

Theorem 7.1 (The classification theorem for fiberwise A_n-spaces)**.** *Let n be a finite positive integer and G a well-pointed A_n-space of the homotopy type of a CW complex. Then, there exists a fiberwise A_n-space $E_n(G) \to M_n(G)$ with fibers A_n-equivalent to G such that the map $[B, M_n(G)] \to \mathcal{E}^{A_n}G(B)$ defined by the correspondence $[f] \mapsto [f^*E_n(G)]$ is bijective for any well-pointed space B of the homotopy type of a connected CW complex.*

Now, we construct the 'associated principal fibration' for $E_n[E] \to M_n[E]$. Again, let G be an A_n-space. For an ex-fibration E over B whose fibers are pointed homotopy equivalent to G, define a space $C_n[E]$ as

$$C_n[E] = \coprod_{b \in M_n[E]} \left\{ f : Q^n h\mathfrak{A}\text{-equivalence}, d^1 f = G, d^0 f = (E_n[E])_b \right\}.$$

In particular, we denote $C_n(G) = C_n[E_1(G)]$. Then there exists a homotopy fibration

$$F_{A_n}G \longrightarrow C_n[E] \longrightarrow M_n[E],$$

where $F_{A_n}G$ is the space consisting of self $Q^n h\mathfrak{A}$-equivalences on G. Similarly to [Tsu12a, Proposition 5.8], the fiber of the Hurewicz fibration $C_n[E] \to C_{n-1}[E]$ is contractible. Then $C_n(G)$ is weakly contractible since $C_1(G) = EHG$ is weakly contractible.

For an A_n-space G, denote the set of equivalence classes of G-framed fiberwise A_n-spaces over a path-connected pointed space B by $\mathcal{E}^{A_n}G(B)_0$. Similarly to [Tsu12a, Proposition 5.2], a pointed map $f : B \to B'$ induces the map $f^* : \mathcal{E}^{A_n}G(B')_0 \to \mathcal{E}^{A_n}G(B)_0$ which depends only on the pointed homotopy class of f. It is our aim of this section to prove the following classification theorem.

Theorem 7.2 (The classification theorem for framed fiberwise A_n-spaces)**.** *Let n be a finite positive integer and G an A_n-space of the homotopy type of a CW complex. Then, there exists a G-framed fiberwise space $E_n(G) \to M_n(G)$ such that the map $[B; M_n(G)]_0 \to \mathcal{E}^{A_n}G(B)_0$ defined by the correspondence $[f] \mapsto [f^*E_n(G)]$ is bijective for any pointed space B of the homotopy type of a connected CW complex.*

Proof. We fix a G-framing of the universal fiberwise A_n-space $E_n(G)$. First, we see that the map $[B, M_n(G)]_0 \to \mathcal{E}^{A_n}G(B)_0$ is surjective. For a G-framed fiberwise A_n-space $E \to B$, by Theorem 7.1, there are maps $f : E \to E_n(G)$ and $\bar{f} : B \to M_n(G)$ such that f covers $\bar{f}$ and f induces a fiberwise $Q^n h\mathfrak{A}$-equivalence $E \to \bar{f}^*M_n(G)$. Take a 2-simplex $\sigma \in \mathcal{R}_2^n(*)$ such that $d^2\sigma : G \to E_*$ is the G-framing of E and $d^0\sigma : E_* \to E_n(G)_*$ is the restriction of f. Considering $d^1\sigma : G \to E_n(G)$ as a point in $C_n(G)$, there is a path from $d^1\sigma$ to the G-framing $G \to E_n(G)_*$ in $C_n(G)$ since $C_n(G)$ is weakly contractible. Then we can take a 2-simplex $\sigma' \in \mathcal{R}_2^n(*)$ such that $d^2\sigma' : G \to E_*$ and $d^1\sigma' : G \to E_n(G)_*$ are the G-framings of E and $E_n(G)$, respectively, because $d^2\sigma'$ is a $Q^n h\mathfrak{A}$-equivalence. The stationary homotopy of $d^2\sigma = d^2\sigma'$ and the homotopy from $d^1\sigma$ to the G-framing $G \to E_n(G)_*$ constructed above extends to a homotopy from σ to σ' of multiplicative functors, where the target space varies in $E_n(G)$ along this homotopy in general. Thus we have a homotopy from $f_* : E_* \to E_n(G)$ to $d^0\sigma'$. This homotopy extends to a homotopy $F : I \times E \to E_n(G)$ from f

covering some homotopy $\bar{F} : I \times B \to M_n(G)$ from $\bar{f}$. The homotopy F induces a fiberwise $Q^n h\mathfrak{A}$-equivalence $I \times E \to \bar{F}^* M_n(G)$ which restricts to the fiberwise $Q^n h\mathfrak{A}$-equivalence $E \to \bar{f}^* E_n(G)$ induced from f on $\{0\} \times E$. By the property of σ', the resulting maps $g = F|_{\{1\}\times E} : E \to E_n(G)$ and $\bar{g} = \bar{F}|_{\{1\}\times B} : B \to M_n(G)$ induces a framed fiberwise $Q^n h\mathfrak{A}$-map $E \to \bar{g}^* M_n(G)$. Then we obtain the equality $[\bar{g}^* E_n(G)] = [E]$ in $\mathcal{E}^{A_n} G(B)_0$.

The injectivity of the map $[B; M_n(G)]_0 \to \mathcal{E}_0^{A_n} G(B)_0$ is proved similarly to [Tsu12a, Proposition 5.6]. The counterpart of [Tsu12a, Lemma 5.3] is stated as follows. □

Lemma 7.3. *Let $E \to B$ be an ex-fibration and $\{m_i\}_{i=2}^n$ and $\{m'_i\}_{i=2}^n$ be fiberwise A_n-forms on E which restrict to the same A_n-form on the fiber over the basepoint. Then the identity map $E \to E$ is a framed fiberwise $Q^n h\mathfrak{A}$-map $(E, \{m_i\}) \to (E, \{m'_i\})$ if and only if $\{m_i\}$ and $\{m'_i\}$ are homotopic as sections of $M_n[E] \to B$ through a homotopy which is constant on the basepoint.*

Similarly to [Tsu12a, Proposition 6.3], we have the following theorem for fiberwise localizations.

Proposition 7.4. *Let G be a path-connected A_n-space. Denote the classifying map of the fiberwise $\mathcal{P}$-localization of $E_n(G)$ by $\lambda : M_n(G) \to M_n(G_{\mathcal{P}})$ as a framed fiberwise A_n-space. Then for a (G-framed) fiberwise A_n-space E over B classified by $\alpha : B \to M_n(G)$, the fiberwise $\mathcal{P}$-localization of E is classified by $\lambda \circ \alpha$. Moreover, if G is homotopy equivalent to a finite complex and $r \geq 2$, then the induced homomorphism $\lambda_* : \pi_r(M_n(G)) \to \pi_r(M_n(G_{\mathcal{P}}))$ is a $\mathcal{P}$-localization of the abelian group $\pi_r(M_n(G))$.*

Remark 7.5. The finiteness condition for G is used as follows. May [May80, Theorem 4.1] proved that $\lambda_* : \pi_r(M_1(G)) \to \pi_r(M_1(G_{\mathcal{P}}))$ is a $\mathcal{P}$-localization when G is a finite complex. By the fibration

$$\Omega^{n-2}\mathrm{Map}_0(G^{\wedge n}, G) \longrightarrow M_n(G) \longrightarrow M_{n-1}(G)$$

and the fact that $\mathrm{Map}_0(G^{\wedge n}, G) \to \mathrm{Map}_0(G_{\mathcal{P}}^{\wedge n}, G_{\mathcal{P}})$ $\mathcal{P}$-localizes each connected components for a connected finite complex G [HMR75, Theorem 3.11], one can show that $\lambda_* : \pi_r(M_n(G)) \to \pi_r(M_n(G_{\mathcal{P}}))$ is a $\mathcal{P}$-localization of the abelian group $\pi_r(M_n(G))$ if G is homotopy equivalent to a finite complex and $r \geq 2$.

8. An application to A_n-types of gauge groups

Let us recall elementary facts on gauge groups. In this section, we assume that every principal bundle have the basepoint in the fiber over the basepoint. For a principal G-bundle P over a pointed finite CW complex B, the *gauge group* $\mathcal{G}(P)$ of P is the topological group consisting of unpointed self bundle maps $P \to P$ covering the identity on B, which is topologized by the compact open topology. Note that $\mathcal{G}$ is a functor from the category of principal bundles and bundle maps to the one of topological groups and continuous homomorphisms. We denote the restriction map on the fiber over the basepoint by $ev : \mathcal{G}(P) \to \mathcal{G}(P|_*) \cong G$ and the kernel of $ev : \mathcal{G}(P) \to G$ by $\mathcal{G}_0(P)$. In other words, the subgroup $\mathcal{G}_0(P) \subset \mathcal{G}(P)$ consists of pointed self bundle maps $P \to P$ covering the identity map on B.

Gottlieb [Got72] has shown that the classifying space $B\mathcal{G}(P)$ of $\mathcal{G}(P)$ has the homotopy type of the connected component $\mathrm{Map}(B, BG; \alpha) \subset \mathrm{Map}(B, BG)$ of the basepoint free mapping space containing the classifying map $\alpha : B \to BG$ of P. Similarly, $B\mathcal{G}_0(P)$ has the homotopy type of the connected component $\mathrm{Map}_0(B, BG; \alpha) \subset \mathrm{Map}_0(B, BG)$ of the pointed mapping space containing α. Moreover, there exists a following homotopy fiber sequence:

$$\mathcal{G}_0(P) \to \mathcal{G}(P) \xrightarrow{ev} G \xrightarrow{\delta} \mathrm{Map}_0(B, BG; \alpha) \to \mathrm{Map}(B, BG; \alpha) \xrightarrow{ev} BG,$$

where $ev : \mathrm{Map}(B, BG; \alpha) \to BG$ is the evaluation map on the basepoint.

Take the fiber bundle $\mathrm{ad}\, P = P \times_G G$ associated to P, where the left G-action on G is given by conjugation. We call $\mathrm{ad}\, P$ the *adjoint group bundle* of P. The adjoint group bundle $\mathrm{ad}\, P$ is a fiberwise topological group. Then the space $\Gamma(\mathrm{ad}\, P)$ of sections is a topological group with the multiplication

is induced from the fiberwise topological group structure on $\mathrm{ad}\, P$. In fact, the gauge group $\mathcal{G}(P)$ is naturally isomorphic to $\Gamma(\mathrm{ad}\, P)$ as a topological group. Note that the basepoint of P defines the natural G-framing $\mathrm{ad}\, P|_* \cong G$ which is an isomorphism of topological groups.

Remark 8.1. The finiteness theorem on fiberwise A_n-types of adjoint group bundles [Tsu12a, Theorem 8.6] also holds for framed fiberwise A_n-types. The proof is almost same as the original one except for we need to use the fibration

$$\Omega_0^{n-2}\mathrm{Map}_0(G^{\wedge n}, G) \longrightarrow M_n(G) \longrightarrow M_{n-1}(G).$$

We shall investigate gauge groups using the localization technique. In general, though a localization of topological group is not a topological group, a localization of an A_n-space has a natural A_n-form by Proposition 2.11. But, as explained before, it is difficult for us to handle the homotopy pullback of general A_n-maps. To avoid this difficulty, we replace the Lie group G by a convenient one.

Proposition 8.2. *Let G be a compact connected Lie group. For a partition $\mathcal{P} \sqcup \mathcal{P}'$ of the set of primes, there exist a $\mathcal{P}$-localization $G_{\mathcal{P}}$, a $\mathcal{P}'$-localization $G_{\mathcal{P}'}$ and a rationalization $G_{(0)}$ of G such that they are topological groups and a rationalization $\rho : G_{\mathcal{P}} \to G_{(0)}$ and $\rho' : G_{\mathcal{P}'} \to G_{(0)}$ can be taken to be homomorphisms as well as Hurewicz fibrations.*

Proof. We may take the localized classifying spaces $(BG)_{\mathcal{P}}$, $(BG)_{\mathcal{P}'}$ and $(BG)_{(0)}$ as countable simplicial complexes. Take rationalizations $\bar{\rho} : (BG)_{\mathcal{P}} \to (BG)_{(0)}$ and $\bar{\rho}' : (BG)_{\mathcal{P}'} \to (BG)_{(0)}$ as simplicial maps. Denote the Milnor's simplicial loop spaces [Mil56] of them by $(G_{\mathcal{P}})_0$, $(G_{\mathcal{P}'})_0$ and $G_{(0)}$, which are topological groups with classifying spaces $(BG)_{\mathcal{P}}$, $(BG)_{\mathcal{P}'}$ and $(BG)_{(0)}$, respectively. Then $\bar{\rho}$ and $\bar{\rho}'$ induce rationalizing homomorphisms $\rho_0 : (G_{\mathcal{P}})_0 \to G_{(0)}$ and $\rho_0' : (G_{\mathcal{P}'})_0 \to G_{(0)}$, respectively. As stated in Remark 4.2, ρ_0 and ρ' can be replaced as Hurewicz fibrations $\rho : G_{\mathcal{P}} \to G_{(0)}$ and $\rho' : G_{\mathcal{P}'} \to G_{(0)}$ which are homomorphisms between topological groups. $\square$

The topological group $\hat{G}$ obtained as the pullback of the diagram $G_{\mathcal{P}} \xrightarrow{\rho} G_{(0)} \xleftarrow{\rho'} G_{\mathcal{P}'}$ is A_∞-equivalent to G and the natural projections $\lambda : \hat{G} \to G_{\mathcal{P}}$ and $\lambda' : \hat{G} \to G_{\mathcal{P}}$ are localizing homomorphisms. Then principal G-bundles and principal $\hat{G}$-bundles correspond one-to-one up to isomorphism and, considering the homotopy types of the classifying spaces, this correspondence preserves the A_∞-types of gauge groups. If P is a principal $\hat{G}$-bundle, then the associated bundles $P_{\mathcal{P}}$, $P_{\mathcal{P}'}$ and $P_{(0)}$ induced from the homomorphisms $\hat{G} \to G_{\mathcal{P}}$, $\hat{G} \to G_{\mathcal{P}'}$ and $\hat{G} \to G_{(0)}$ give the corresponding fiberwise localizations $\mathrm{ad}\, P_{\mathcal{P}}$, $\mathrm{ad}\, P_{\mathcal{P}'}$ and $\mathrm{ad}\, P_{(0)}$ of $\mathrm{ad}\, P$, respectively.

Now we concentrate on our attention to the gauge groups of principal G-bundles over the r-dimensional sphere S^r. Fix a pointed map $\epsilon : S^{r-1} \to G$. For an integer $k \in \boldsymbol{Z}$, we denote the principal G-bundle with classifying map $k\epsilon$ by P_k. As easily checked, if the adjoint group bundles $\mathrm{ad}\, P_k$ and $\mathrm{ad}\, P_{k'}$ are G-framed fiberwise A_n-equivalent, then one can deform the framed fiberwise A_n-equivalence $\mathrm{ad}\, P_k \to \mathrm{ad}\, P_{k'}$ to the one $f : \mathrm{ad}\, P_k \to \mathrm{ad}\, P_{k'}$ which restricts to the isomorphism of topological groups on the fiber over the basepoint since the framings of adjoint group bundles are given by isomorphisms of topological groups. Moreover, if $\varphi : D^r \to S^r$ is the characteristic map, then we obtain the following strictly commutative diagram of topological groups:

$$\begin{tikzcd}
\Gamma(\varphi^*\mathrm{ad}\, P_k) \arrow[r] \arrow[d, "(\varphi^* f)_*"'] & \Gamma^{\mathrm{id}}((\varphi^*\mathrm{ad}\, P_k)|_{S^{r-1}}) \arrow[d, "(\varphi^* f)_*"] & G \arrow[l] \arrow[d, equal] \\
\Gamma(\varphi^*\mathrm{ad}\, P_{k'}) \arrow[r] & \Gamma^{\mathrm{id}}((\varphi^*\mathrm{ad}\, P_{k'})|_{S^{r-1}}) & G \arrow[l]
\end{tikzcd}$$

where $\Gamma^{\mathrm{id}}((\varphi^*\mathrm{ad}\, P_k)|_{S^{r-1}})$ represents the identity component of $\Gamma((\varphi^*\mathrm{ad}\, P_k)|_{S^{r-1}})$, $\Gamma(\varphi^*\mathrm{ad}\, P_k) \to \Gamma^{\mathrm{id}}((\varphi^*\mathrm{ad}\, P_k)|_{S^{r-1}})$ and $\Gamma(\varphi^*\mathrm{ad}\, P_{k'}) \to \Gamma^{\mathrm{id}}((\varphi^*\mathrm{ad}\, P_{k'})|_{S^{r-1}})$ are the restrictions on the boundary S^{r-1} of D^r, which are Hurewicz fibrations, $G \to \Gamma^{\mathrm{id}}((\varphi^*\mathrm{ad}\, P_k)|_{S^{r-1}})$ and $G \to \Gamma^{\mathrm{id}}((\varphi^*\mathrm{ad}\, P_{k'})|_{S^{r-1}})$ are the inclusions through the framings of the fiber over the basepoint and $\Gamma(\varphi^*\mathrm{ad}\, P_k) \to \Gamma(\varphi^*\mathrm{ad}\, P_{k'})$ and

$\Gamma((\varphi^*\mathrm{ad}\,P_k)|_{S^{r-1}}) \to \Gamma((\varphi^*\mathrm{ad}\,P_{k'})|_{S^{r-1}})$ are the induced A_n-equivalence from the framed fiberwise A_n-equivalence $\varphi^* f : \varphi^*\mathrm{ad}\,P_k \to \varphi^*\mathrm{ad}\,P_{k'}$. The gauge groups $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are isomorphic to the pullback of the corresponding horizontal lines. Since $\varphi^* P_k$ and $\varphi^* P_{k'}$ are trivial bundles, the following diagram is equivalent to the above one,

$$\begin{tikzcd}
\mathrm{Map}\,(D^r, G) \arrow[r] \arrow[d, equal] & \mathrm{Map}\,(S^{r-1}, G; 0) \arrow[d, "F"] & G \arrow[l, "\alpha_k"'] \arrow[d, equal] \\
\mathrm{Map}\,(D^r, G) \arrow[r] & \mathrm{Map}\,(S^{r-1}, G; 0) & G \arrow[l, "\alpha_{k'}"']
\end{tikzcd}$$

where $\mathrm{Map}\,(S^{r-1}, G; 0)$ denotes the identity components of $\mathrm{Map}\,(S^{r-1}, G)$. The map $\alpha_k : G \to \mathrm{Map}\,(S^{r-1}, G; 0)$ is defined as $\alpha_k(g)(s) = (k\epsilon)(s)g(k\epsilon)(s)^{-1}$. The right square commutes strictly and the left one up to homotopy of A_n-maps. Denote this homotopy by H. Since F is induced from the framed fiberwise A_n-equivalence f, the composite of H and the evaluation map is the stationary homotopy of the evaluation map $\mathrm{Map}\,(D^r, G) \to G$. Now, we give the proof of our main theorem.

Proof of Theorem 1.1. Let $\mathcal{P}$ be the set of primes which divide N and $\mathcal{P}'$ the set of other primes. As above, we may assume that the structure group is $\hat{G}$. Denote the localizations by $\lambda : \hat{G} \to G_{\mathcal{P}}$, $\lambda' : G \to G_{\mathcal{P}'}$, $\rho : G_{\mathcal{P}} \to G_{(0)}$ and $\rho' : G_{\mathcal{P}'} \to G_{(0)}$. So, in the proof of this theorem, we denote P_k the principal $\hat{G}$-bundle corresponding to the original principal G-bundle P_k, and so on. Then, since $(N, k) = (N, k')$, there exists an integer $A \in \boldsymbol{Z}$ prime to N such that $Ak \equiv k' \bmod N$. From the classification theorem of $\hat{G}$-framed fiberwise A_n-spaces, $\mathrm{ad}\,P_{Ak}$ and $\mathrm{ad}\,P_{k'}$ are framed fiberwise A_n-equivalent. Thus we have the following diagram:

(*)
$$\begin{tikzcd}
\mathrm{Map}\,(D^r, G_{\mathcal{P}}) \arrow[r] \arrow[d, equal] & \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}}; 0) \arrow[d, "A^*"] & G_{\mathcal{P}} \arrow[l, "\alpha_k"'] \arrow[d, equal] \\
\mathrm{Map}\,(D^r, G_{\mathcal{P}}) \arrow[r] \arrow[d, equal] & \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}}; 0) \arrow[d, "F"] & G_{\mathcal{P}} \arrow[l, "\alpha_{Ak}"'] \arrow[d, equal] \\
\mathrm{Map}\,(D^r, G_{\mathcal{P}}) \arrow[r] & \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}}; 0) & G_{\mathcal{P}} \arrow[l, "\alpha_{k'}"']
\end{tikzcd}$$

where F is an A_n-equivalence induced from the framed fiberwise A_n-equivalence $\mathrm{ad}\,P_{Ak} \to \mathrm{ad}\,P_{k'}$ which restricts to an isomorphism on the fiber over the basepoint, the right squares commute strictly and the left ones up to homotopies such that the composite of them with the evaluation maps are stationary homotopy. We remark that the map $A^* : \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}}; 0) \to \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}}; 0)$ is an A_∞-equivalence since A is not divided by the primes in $\mathcal{P}$. Similarly, since $\mathrm{ad}\,P_{Ak,\mathcal{P}'}$ and $\mathrm{ad}\,P_{k',\mathcal{P}'}$ are A_n-trivial by Proposition 7.4, we obtain the following diagram:

(**)
$$\begin{tikzcd}
\mathrm{Map}\,(D^r, G_{\mathcal{P}'}) \arrow[r] \arrow[d, equal] & \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}'}; 0) \arrow[d, "F'"] & G_{\mathcal{P}'} \arrow[l, "\alpha_k"'] \arrow[d, equal] \\
\mathrm{Map}\,(D^r, G_{\mathcal{P}'}) \arrow[r] & \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}'}; 0) & G_{\mathcal{P}'} \arrow[l, "\alpha_{k'}"']
\end{tikzcd}$$

where F' is an A_n-equivalence, the right square commutes strictly and the left one commutes up to homotopy with the same property as stated above. Note that the evaluation map $\mathrm{Map}\,(S^{r-1}, G_{(0)}; 0) \to G_{(0)}$ is a homotopy equivalence because $\Omega_0^{r-1} G_{(0)}$ is contractible since $r \geq 2n_\ell$ and $\Omega_0^{r-1} G_{(0)} \to \mathrm{Map}\,(S^{r-1}, G_{(0)}; 0) \to G_{(0)}$ is a fibration. Then, since the pullback of the diagram $\mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}}; 0) \xrightarrow{\rho_*} \mathrm{Map}\,(S^{r-1}, G_{(0)}; 0) \xleftarrow{\rho'_*} \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}'}; 0)$ is $\mathrm{Map}\,(S^{r-1}, \hat{G}; 0)$, there exists a lift $\hat{F} : \mathrm{Map}\,(S^{r-1}, \hat{G}; 0) \to \mathrm{Map}\,(S^{r-1}, \hat{G}; 0)$, which is an A_n-equivalence, of $F \circ A^* \circ \lambda_* : \mathrm{Map}\,(S^{r-1}, \hat{G}; 0) \to \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}}; 0)$

and $F' \circ \lambda'_* : \mathrm{Map}\,(S^{r-1}, \hat{G}; 0) \to \mathrm{Map}\,(S^{r-1}, G_{\mathcal{P}'}; 0)$ by Theorem 3.1. This lift is determined uniquely up to homotopy by Theorem 3.3 taking the homotopy between $\rho_* \circ F \circ A^* \circ \lambda_*$ and $\rho'_* \circ F' \circ \lambda'_*$ as the lift of the stationary homotopy of $ev \circ \rho_* \circ \lambda_* : \mathrm{Map}\,(S^{r-1}, \hat{G}; 0) \to G_{(0)}$ through the homotopy equivalence $ev : \mathrm{Map}\,(S^{r-1}, G_{(0)}) \to G_{(0)}$. Now we have the following diagram:

$$\begin{tikzcd}
\mathrm{Map}\,(D^r, \hat{G}) \arrow[r] \arrow[d, equal] & \mathrm{Map}\,(S^{r-1}, \hat{G}; 0) \arrow[d, "\hat{F}"] & \hat{G} \arrow[l, "\alpha_k"'] \arrow[d, equal] \\
\mathrm{Map}\,(D^r, \hat{G}) \arrow[r] & \mathrm{Map}\,(S^{r-1}, \hat{G}; 0) & \hat{G} \arrow[l, "\alpha_{k'}"']
\end{tikzcd}$$

Again, using Theorem 3.1 and 3.3, this diagram commutes up to homotopy of A_n-maps. Therefore, taking the pullback along the horizontal direction, $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n-equivalent by Theorem 3.1. □

Now, let us consider the gauge groups of principal SU(2)-bundles over S^4. Denote the principal SU(2)-bundle over S^4 of the second Chern number $k \in \boldsymbol{Z}$ by P_k, where this notation is compatible with the notation in Theorem 1.1 for the appropriate generator $\epsilon \in \pi_3(\mathrm{SU}(2)) \cong \boldsymbol{Z}$. We estimate the divisibility of the least positive number a_n^{fw} such that the adjoint group bundle $\mathrm{ad}\,P_{a_n^{\mathrm{fw}}}$ is A_n-trivial. From the result of [CS00], we have $a_1^{\mathrm{fw}} = 12$ and $a_2^{\mathrm{fw}} = 180$. By Theorem 1.1, if the equality $(k, a_n^{\mathrm{fw}}) = (k', a_n^{\mathrm{fw}})$ holds, then $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n-equivalent. We define the following A_n-equivalence invariant of $\mathcal{G}(P_k)$:

$$d_p(k) = \max\{\, n \in \boldsymbol{Z}_{\geq 0} \mid \mathcal{G}^{\mathrm{id}}(P_k)_{(p)} \text{ is } A_n\text{-equivalent to } \mathcal{G}^{\mathrm{id}}(P_0)_{(p)} \,\},$$

where $\mathcal{G}^{\mathrm{id}}(P_k)$ denotes the identity component of $\mathcal{G}(P_k)$. This $d_p(k)$ will turn out to be equal to the one defined by [Tsu01] by the result of [KK10] and the following proposition.

Proposition 8.3 (Proposition 10.2 of [Tsu12a])**.** *The localized gauge group $\mathcal{G}^{\mathrm{id}}(P_k)_{(p)}$ is A_n-equivalent to $\mathcal{G}^{\mathrm{id}}(P_0)_{(p)}$ if and only if the fiberwise localized adjoint group bundle* $\mathrm{aut}\,P_{k,p}$ *is A_n-trivial.*

Remark 8.4. The following is immediately follows from this proposition:

$$d_p(k) = \max\{\, n \in \boldsymbol{Z}_{\geq 0} \mid v_p(a_n^{\mathrm{fw}}) \leq v_p(k) \,\}.$$

By the result of [Tsu12b, Section 5], we have $d_p(pk) > d_p(k)$ for an odd prime p and $d_2(4k) > d_2(k)$. Thus we obtain the following inequalities:

$$\begin{aligned}
&0 \leq v_2(a_{n+1}^{\mathrm{fw}}) - v_2(a_n^{\mathrm{fw}}) \leq 2, \qquad v_2(a_n^{\mathrm{fw}}) \leq 2n, \\
&0 \leq v_p(a_{n+1}^{\mathrm{fw}}) - v_p(a_n^{\mathrm{fw}}) \leq 1, \qquad v_p(a_n^{\mathrm{fw}}) \leq n \quad (p : \mathrm{odd}).
\end{aligned}$$

In [Tsu12b], the author has shown the following result ([Tsu12b, Proposition 4.4] is correct but [Tsu12b, Theorem 4.5] is not correct for an odd prime p):

$$d_2(k) \leq v_2(k), \qquad d_p(k) \leq \frac{p-1}{2}(v_p(k)+1) - 1 \quad (p : \mathrm{odd}).$$

Since we have $d_p(a_n^{\mathrm{fw}}) \geq n$ by definition of a_n^{fw}, we obtain

$$v_2(a_n^{\mathrm{fw}}) \geq n, \qquad v_p(a_n^{\mathrm{fw}}) \geq \frac{2(n+1)}{p-1} - 1 \quad (p : \mathrm{odd}).$$

From these, we have the next proposition.

Proposition 8.5. *The following inequalities hold:*

$$n \leq v_2(a_n^{\mathrm{fw}}) \leq 2n, \qquad \left[\frac{2n}{p-1}\right] \leq v_p(a_n^{\mathrm{fw}}) \leq n \quad (p : \mathrm{odd}),$$

where $[a]$ denotes the largest integer which is not larger than a.

Remark 8.6. In particular, we have $v_3(a_n^{\mathrm{fw}}) = n$.

Moreover, the homotopy groups are known to be

$$\pi_{4n+3}(\boldsymbol{H}P^\infty_{(p)}) = \begin{cases} \boldsymbol{Z}/p\boldsymbol{Z} & (n \equiv 0 \bmod (p-1)/2) \\ 0 & (\text{otherwise}) \end{cases}$$

for $p \geq 5$ and $1 \leq n < (2p+1)(p-1)/2 - 1$ by the result of [Tod65]. Observing the obstruction to extending the composite

$$S^4 \vee \boldsymbol{H}P^n \xrightarrow{k \vee (\text{inclusion})} \boldsymbol{H}P^\infty \vee \boldsymbol{H}P^\infty \xrightarrow{(\text{folding map})} \boldsymbol{H}P^\infty \xrightarrow{(\text{localization})} \boldsymbol{H}P^\infty_{(p)}$$

over $S^4 \times \boldsymbol{H}P^n$ as in the argument of [Tsu12b, Section 5] ($\mathrm{ad}\, P_k$ is A_n-trivial if and only if it extends over $S^4 \times \boldsymbol{H}P^n$ by [KK10]), one can see that

$$v_p(a_n^{\mathrm{fw}}) = \left[\frac{2n}{p-1} \right]$$

for $1 \leq n < (2p+1)(p-1)/2 - 1$.

We show a necessary condition for that $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n-equivalent.

Proposition 8.7. *For an odd prime p, if $\mathcal{G}^{\mathrm{id}}(P_k)_{(p)}$ and $\mathcal{G}^{\mathrm{id}}(P_{k'})_{(p)}$ are A_n-equivalent, then* $\min\{v_p(a_n^{\mathrm{fw}}), v_p(k)\} = \min\{v_p(a_n^{\mathrm{fw}}), v_p(k')\}$.

Proof. If $\mathcal{G}^{\mathrm{id}}(P_k)_{(p)}$ and $\mathcal{G}^{\mathrm{id}}(P_{k'})_{(p)}$ are A_n-equivalent, then we have $\max\{n, d_p(k)\} = \max\{n, d_p(k')\}$. Since $d_p(k) = \max\{\, m \in \boldsymbol{Z}_{\geq 0} \mid v_p(k) \geq v_p(a_m^{\mathrm{fw}}) \,\}$ and $0 \leq v_p(a_{m+1}^{\mathrm{fw}}) - v_p(a_m^{\mathrm{fw}}) \leq 1$ for an odd prime p, we have $\max\{v_p(a_n^{\mathrm{fw}}), v_p(k)\} = \max\{v_p(a_n^{\mathrm{fw}}), v_p(k')\}$. $\square$

Remark 8.8. If we set $p = 2$ and $n \leq 2$, this proposition remains true by the results of Kono [Kon91] and of Crabb and Sutherland [CS00].

As a corollary of this proposition, Theorem 1.2 immediately follows.

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